Polly Want a Nomial

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Monday Mathematics #5

Factor: $x^3 + 5x^2 - 89x + 195$

While some other topics come up often enough that you need to know about them, polynomials come up so often that you can hardly do anything *without* them.

Everyone's all anxious to hear how to factor polynomials, but answering that question directly isn't very helpful. Factoring them is a lot easier if you know some basic theorems. But first, terminology.

ax^n is a monomial, where x is a variable, a and n are constants, and n is a whole number. When you add monomials together, you get polynomials. The highest value of n is called the degree of a polynomial. If the polynomial has multiple variables, you add their exponents to get the degree. (The degree of $6x^2yz$ is 4.) Polynomials of degree zero are called constants; of one, linear; two, quadratic; three, cubic.

A polynomial, as you have noticed, is an expression and not an equation. The values of x that make it equal zero are called the roots (or zeroes) of the polynomial. When we have polynomials, we are usually concerned about its roots. A factored polynomial takes the form $(x - r_1)(x - r_2)(x - r_3)\ldots$ where the rs are the roots, but you probably knew that already.

Not surprisingly, the most important theorem about polynomials is the Fundamental Theorem of Algebra. "Every polynomial of degree n has at most n distinct complex roots." For example, all quadratics have one or two complex roots, though none of them might be real. (Actually, each polynomial has *exactly* n complex roots, though some of these might be equal to each other. We call that phenomenon "multiplicity.")

So now we know something about how many roots the polynomial has. We can find out more using Descartes's Rule of Signs.

Descartes said, "Given a polynomial $P(x)$, the number of positive roots is the number of sign changes in the coefficients of $P(x)$ minus an even number greater than or equal to 0. The number of negative roots is the number of sign changes in the coefficients of $P(-x)$ minus an even number greater than or equal to 0."

What does that mean? The coefficients of $P(x)$ are 1, 5, –89, and 195. There are two sign changes: + to – from 5 to –89, then – to + from –89 to 195. Now we know there are either two or zero positive roots.

The coefficients of $P(-x)$ can be found by taking the opposite of all the odd-degree coefficients. This leaves us with –1, 5, 89, and 195. There's only one sign change here, so we know there is exactly one negative root. See how much we know already? And we haven't even started to try factoring it.

Now we can use our most powerful tool, Vieta's formula. But first, the following notation:

$$\begin{aligned} s_1 &= r_1 + r_2 + r_3 \\ s_2 &= r_1 r_2 + r_1 r_3 + r_2 r_3 \\ s_3 &= r_1 r_2 r_3 \end{aligned}$$

Basically, s_1 is the sum of all roots multiplied one at a time, s_2 is the sum of all roots pairwise multiplied, and s_3 is the product of all roots. Extending the notation to non-cubic polynomials should be straightforward enough.

So here it is, your secret weapon: Vieta's formula.

$$s_i = (-1)^i \cdot \frac{a_{d-i}}{a_d}$$

Where d is the degree of the polynomial, and a_n is the coefficient of the n-degree term.

So the sum of our roots is:

$$\begin{aligned} s_1 &= (-1)^1 \cdot \frac{a_{3-1}}{a_3} \\ s_1 &= -1 \cdot \frac{5}{1} = -5 \end{aligned}$$

The product, likewise, is –195. (In general, the sum of the roots of a polynomial is the opposite of the second coefficient over the first. The product of the roots of an even-degree polynomial is the last coefficient over the first, and the opposite of that for an odd-degree polynomial.)

Now, clever you finds the prime factorization of –195, which is $-1 \cdot 3 \cdot 5 \cdot 13$. Hey look! Three terms, one of which is negative. You already know that only one (not three) is negative, thanks to Descartes's Rule of Signs.

Instead of blindly guessing factors, you plug in –13, and the polynomial equals zero. The other two factors are 3 and 5, and you have the factorization:

$$(x-3)(x-5)(x+13)$$

So before you start trying to factor by guessing, remember the sum and product of the roots. A good general approach would be to factor out 1 and –1 as many times as possible, and then factor the product of the roots. The remaining roots must be either prime, or composite products of those prime numbers, so you have a finite number of integers to check.

Unless, of course, the roots aren't all integral or real. But then you'll probably have a calculator.

All you ever really needed to know to factor polynomials intelligently: the Fundamental Theorem of Algebra, Descartes's Rule of Signs, and especially Vieta's formulas. (Most important are the sum and product cases, so memorize those even if you don't memorize Vieta's generalized formula.)

And until next time, remember: polynomials are your friends. Your lifeless, inky, mathematical friends. I really hope your other friends make for better conversation.