

Perfectly Log-ical

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Monday Mathematics #4

Simplify completely: $(\log_4 27)(\log_9 625)(\log_5 16)$

How many digits are in the base 7 representation of 235, 654, 123, 639?

Two problems this time.¹ Though they might seem completely different, in reality... well, okay, they *are* completely different. But they do have one thing in common.

Hopefully you remember logarithms are the inverse operation of exponentiation. The log base b of a is x if x is the power to which b is raised to get a .

$$\log_b a = x \iff a = b^x$$

If you have trouble remembering that, my helpful hint is that b is the “base” of both the logarithm and the exponentiation process, and that a logarithm evaluates to an exponent. If you like memory devices, a useless mnemonic for the previous sentence is BITBOBTLATEP ALETAE, the former sounding like an Egyptian pharaoh, and the latter like a species of plant.

Some helpful properties of logarithms follow:

$$\begin{aligned}\log_b x + \log_b y &= \log_b xy \\ \log_b x - \log_b y &= \log_b \frac{x}{y} \\ \log_b x &= \frac{\log_a x}{\log_a b}\end{aligned}$$

On some calculators, the change-of-base formula above is the only way to calculate logarithms other than the common (base 10) and natural (base e) logarithms.

¹First problem is from an ARML practice packet.

We know all we need to solve the first problem now. No calculators allowed. Just use change-of-base repeatedly to reduce the expression into a single logarithm. First, we recognize that 27 is a power of 3, so the first logarithm would be nicer with a base of 3.

$$\frac{(\log_3 27)(\log_9 625)(\log_5 16)}{(\log_3 4)}$$

We can also rewrite the other two logarithms in base 5 and 4, respectively.

$$\frac{(\log_3 27)(\log_5 625)(\log_4 16)}{(\log_3 4)(\log_5 9)(\log_4 5)}$$

Three needs to be raised to the third power to get 27, so we know (from the definition of a logarithm) that $\log_3 27$ is 3. By the same token, $\log_5 625$ is 4, and $\log_4 16$ is 2.

$$\frac{3 \cdot 4 \cdot 2}{(\log_3 4)(\log_5 9)(\log_4 5)}$$

At least the numbers are smaller. Now apply change-of-base on the denominator, flipping its denominator onto the numerator.

$$\frac{(24)(\log_4 3)(\log_3 5)(\log_5 4)}{(\log_4 4)(\log_3 9)(\log_5 5)}$$

Simplify.

$$\frac{(24)(\log_4 3)(\log_3 5)(\log_5 4)}{1 \cdot 2 \cdot 1}$$

This problem is looking better each time. Now, we see that we can start cancelling things if we convert the middle logarithm to base 4.

$$\frac{(12)(\log_4 3)(\log_4 5)(\log_5 4)}{(\log_4 3)}$$

Cancel.

$$(12)(\log_4 5)(\log_5 4)$$

And do it again.

$$\frac{(12)(\log_5 5)(\log_5 4)}{(\log_5 4)}$$

And, well, simplify.

$$12$$

Just because you couldn't evaluate any of the terms by hand doesn't mean you couldn't evaluate their product by hand. The hint to use change-of-base came from the problem itself, because all the numbers were integral powers of 3, 4, and 5, except that they were with the wrong logarithmic base. So we converted the bases, and everything kept dropping out.

Now for the second problem. The obvious way to do this problem is to convert into base 7, which is rather painful with or without a calculator. (Unless you have a program to do it, but that's still not the easiest way.)

What could logarithms have to do with this problem? You think back to number bases, and suddenly remember that all the place values are powers of the radix. The only place values used are ones that go into the number at least once. So you could solve this problem fairly quickly by calculating all the powers of seven, and finding the first power larger than the number at hand.

Better yet, just take the logarithm, base 7, of that big number. (Use change-of-base if necessary on your calculator.) The answer is about 13.457. That means that 7^{13} is the last place value used in the number, because the number is larger than 7^{13} but smaller than 7^{14} . So the place values used are $7^0, 7^1, 7^2 \dots$ up to 7^{13} . That's 14 place values, 14 digits, and the answer 14, without ever finding a single digit of the base 7 representation. (Which is, by the way, 23011505000224_7 , in all its fourteen digits of glory.)

Logarithms! Make sure you know the basic identities (so you can solve logarithm problems, like #1) and the definition (so you can even use them in problems that don't mention logarithms, like #2).

Well, I could go on forever, but I'm going now to look for some *B. aletae* flowers in my backyard—I hear they make good herbal tea. And until next time, remember: the opposite of exponent isn't tnenopxe, it's logarithm.

King Bitbobtlatep would approve.