All Your Base

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Monday Mathematics #3

Convert 52.73_{10} into binary, truncating at two decimal places.

While you've gotta know how to deal with repeating equations, you've *really* gotta know how to manipulate numbers in different number bases.

First, let's look at base 10 or decimal, the number base we use every day. 52.73_{10} is a perfectly ordinary decimal number. (The 10 subscript means that it's in base 10.) What does it represent? Let's go back to the discussion of "place values" in your elementary school class...

$$52.73 = 5 \cdot 100 + 2 \cdot 10 + 1 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100}$$

The "tens" place holds as many whole tens there are, then the "ones" place holds as many whole ones there are, then the decimal point, then the tenths, then the hundredths.

Now the same, from a base perspective:

$$52.73_{10} = 5 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^{-1} + 3 \cdot 10^{-2}$$

The first place left of the decimal point holds the number of whole 10^1s are available, and the place to the left of that holds as many 10^2s are available. Going the other direction, the first place right of the decimal point holds 10^{-1}s, and the place to the right of that holds 10^{-2}s. In other words,

$$AB.CD_b = A \cdot b^2 + B \cdot b^1 + C \cdot b^{-1} + D \cdot b^{-2}$$

Okay then. Now let's try binary. What's the largest place value we'll need? What *are* the place values? (The powers of two, of course.) What's the largest power of two that goes into 52.73? 1, 2, 4, 8, 16, 32, 64…32, then. That's 2^5. It goes in once, so we subtract a 32 from 52.73, and we're left with 20.73.

$$1 \cdot 2^5 + 20.73$$

Now one 16 goes in, so we subtract that out.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 4.73$$

No 8s go in.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 4.73$$

But a 4 does.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0.73$$

And no 2s or 1s do.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 0.73$$

Since we're left with a fractional part less than 1 we have to continue to the right of the decimal point. (Remember, 1 is the same in every base.) We raise two to negative exponents now. 1/2, 1/4, 1/8… well, 1/2 goes in once.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1} + 0.23$$

But 1/4 doesn't.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 0.23$$

We could keep going, but the question said to stop at two decimal places. Now, we just need to represent our base 2 number in base 2.

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1} + 0 \cdot 2^{-2} = 110100.10_2$$

And we're done! The answer is 110100.10_2. You could convert this back, but I leave that as an exercise for the reader. Notice, though, that we only needed the digits 0 and 1, but in base 10 we need the digits 0 through 9. In general, any base b requires only the digits 0 through $b - 1$.

Notice also that a number only has one representation in any given base. Each number base provides a different representation of a number, but a representation unique in that base.

And until next time, remember: in Soviet Russia, base converts *you*.