It Keeps Going. . . And Going. . .

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Monday Mathematics #2

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

There are 10_2 types of people in the world, those that have to stop and think when they see that problem, and those that have already seen it and know how to solve all problems like it forever after. This one's a classic, folks.

The key is that it repeats infinitely. Each radical is equal to each other radical, because they all go on forever—the *same* forever. It might sound obvious at first, but it's a really important point that many people refuse to accept. They all have the same number of radicals underneath them: the first, the second, the tenth. The second *doesn't* have one less than the first. They both have an infinite number. The "one less" isn't only insignificant, it's nonexistent.

Once you've convinced yourself of that fact, it should be pretty apparent what to do. The second square root is the same as the first square root, which is the same as x. Substitute the infinite quantity for the finite one, and the problem becomes trivial.

$$x = \sqrt{1 + x}$$

The solution is, of course, the positive root of $x^2 = 1+x$. The extra bonus of the day is that this happens to be $(\sqrt{5}+1)/2$, the magically omnipresent golden ratio ϕ.

The same method can be applied to other famous problems.

$$.999999\ldots = \ ?$$

So *is* .9 repeating equal to 1? If you accept my solution to the first problem, you're not allowed to argue with the next solution:

$$\begin{aligned}
x &= .9999\ldots \\
10x &= 9.9999\ldots \\
10x - x &= 9.9999\ldots - .9999\ldots \\
9x &= 9 \\
x &= 1 \\
.9999\ldots &= 1
\end{aligned}$$

Believe it.

The arguments come when people say that $9.\bar{9}$ has one less 9 at the end than $.\bar{9}$. Infinity isn't just a really large number. It's an infinitely large number. You take an apple out of the infinite barrel, and nobody even notices. Not even the barrel. You didn't change it at all.

If you have the feeling like the previous proof is pulling a fast one, of course, you can simply recall the formulas of an infinite geometric series.

$$\begin{aligned}
a_n &= a_0 r^n \\
\sum_{i=0}^{\infty} a_i &= \frac{a}{1-r}
\end{aligned}$$

$.\bar{9}$ is just an infinite geometric series, really. It's .9, plus .9 times .1, plus .9 times .01, plus .9 times .001...

$$\begin{aligned}
a_n &= .9 \cdot .1^n \\
\sum_{i=0}^{\infty}(.9 \cdot .1^i) &= \frac{.9}{1-.1} \\
\sum_{i=0}^{\infty}(.9 \cdot .1^i) &= \frac{.9}{.9}
\end{aligned}$$

It's still 1. What are the chances! (Pretty high, actually.)

Hopefully by now you're a believer of infinity. Those dots don't just mean that it goes on for a really long time. The pattern goes on so long that it never even ends. Even if you skip the first ten terms, it's just as long as it was. Even if you skip all the even terms, it's still no shorter.

So remember, repeating equations are your friend. Try substituting finite terms in place of infinitely-repeating ones. If that doesn't work, see if it's a geometric series. (Geometric series converge iff $|r| < 1$ or $a_0 = 0$.)

And until next time, remember: $.\bar{9}$ really *is* equal to 1.