## Zero Determinants

If a determinant is zero, that means no $n$-volume is spanned by those vectors in $n$-space. This could be the case if you have too few vectors. For example, one vector in two-space contains no area; two vectors in three-space contains no any volume. The same is true if you have too many vectors: three vectors in two-space still contain no area, just as four vectors in three-space have no volume.

In general, the determinant is always zero unless you have exactly $n$ vectors of $n$ elements each. Consequently, only square matrices may have non-zero determinants. (Determinants for non-square matrices are either defined as zero, or considered undefined. I choose the former.)

However, there are circumstances in which $n$ vectors in $n$-space still have no contents. If any of the vectors is the zero vector, no volume is contained. Likewise, if one vector is a scalar multiple of another vector, the box they form collapses, and has no volume. In general, if the vectors are not linearly independent—you guessed it—they form a $(n-1)$-box in $n$-space which still has no $n$-volume. (For example, consider the vectors $i$, $j$, and $i+j$ in three-space, which form a triangle with area, but no volume.)

In fact, when the determinant is zero, that means that the image spanned by the image basis vectors has one less dimension than the number of elements they contain. The image is flatter than the space that contains it—the vectors are not linearly independent. This means that the transformation is a projection onto a lower-dimensional space: there is no inverse. (Think about the preimage box turning into an image pancake.)

## Summary of Properties of Zero Determinants

If, for $n \times n$ matrix $A$, $\det(A) = 0$, then $A$ is singular (non-invertible). Its column vectors are not linearly independent, either because one is the zero vector, or one can be written as a linear combination of the others.

I suppose we could also view $Ax = b$ not only as a transformation, but as a system of linear equations with $n$ equations and $n$ unknowns. If $A$ has a non-zero determinant, there is exactly one preimage $x$ for every image vector $b$. (There is exactly one solution for every $b$.) If $A$ has a zero determinant, $A$ is a projection, so each image vector is mapped to by an infinite number of preimage vectors. (There are an infinite number of solutions for every $b$.) Note that a system with $n$ equations and $n$ unknowns can never be inconsistent: there are always one or infinite solutions.