Orthogonal Complements

For inner product space V and subspace $W \subset V$, recall the orthogonal complement of W is defined as $W^{\perp} = \{\vec{v} \in V : \forall\ \vec{w} \in W,\ \langle \vec{v}, \vec{w} \rangle = 0\}$.

If V has dimension n and W has dimension $r \leq n$, there exists a basis of n vectors for V such that r of those vectors span W. If these vectors are ordered such that the r vectors come first, performing the Gram-Schmidt process on all n vectors in order will yield n orthonormal vectors, of which the first r vectors span W, and the next $n-r$ vectors span another space with dimension $n-r$. Since its basis vectors are all orthogonal to the basis vectors of W, every vector in the latter is orthogonal to every vector in W (why?), making this space a subset of $W^{\perp}$. Therefore, because $r + (n-r) = n$,

$$\dim(W) + \dim(W^{\perp}) = \dim(V)$$

Results!

Recall that, under the Euclidean inner product, the row space and nullspace of a matrix are orthogonal complements. Combining the rank-nullity theorem and orthogonal complement theorem, it is trivial to show that the row space and column space have the same dimension:

$$\dim(r.s.) + \dim(n.s.) = n$$

$$\dim(c.s.) + \dim(n.s.) = n$$

$$\dim(r.s.) = \dim(c.s.)$$

The Importance of Conceptual Thinking

Prove $rank(AB) \leq rank(A)$. (Recall our earlier discussion on dimension.)