Rank and Nullity

Recall ker $T = \{\vec{u} \in V : T(\vec{u}) = 0\}$. Since $T(\vec{u}) = Mu$, if ker T is the set of vectors $\vec{u}$ such that $Mu = 0$, then the kernel of T is equivalent to the nullspace of M. The nullity of M is thus dim(ker T).

Likewise, since vectors $T(V_i)$ span im T, and $T(V_i)$ form the column vectors of M, the column vectors of M span im T. We have already defined the column space of M as the space spanned by its column vectors; therefore, the image of T is equivalent to the column space of M. The rank of M is thus dim(im T).

Rank-Nullity Theorem

Recall that the rank and nullity of a matrix sum to its number of columns. In our new terminology, this can be expressed equivalently as:

$$T : V \to W, \quad \dim(\text{im } T) + \dim(\ker T) = \dim(V)$$

This fact can be proved by constructing a basis for V out of the kernel's basis and the preimage of W's basis. To show this set is indeed a basis, we must prove spanning and linear independence.

$$T : V \to W$$

$$\dim(\text{im } T) = n,\ \text{span}(\vec{w}_1 \ldots \vec{w}_n) = \text{im } T$$

$$\dim(\ker T) = k,\ \text{span}(\vec{u}_1 \ldots \vec{u}_k) = \ker T$$

$$\vec{w}_i = T(\vec{v}_i)$$

$$\vec{v} \in V,\ T(\vec{v}) = b_1\vec{w}_1 + \ldots + b_n\vec{w}_n$$

$$T(b_1\vec{v}_1 + \ldots + b_n\vec{v}_n - \vec{v}) = 0$$

$$T(b_1\vec{v}_1 + \ldots + b_n\vec{v}_n - \vec{v}) = T(a_1\vec{u}_1 + \ldots + a_k\vec{u}_k)$$

$$b_1\vec{v}_1 + \ldots + b_n\vec{v}_n - \vec{v} = a_1\vec{u}_1 + \ldots + a_k\vec{u}_k$$

$$\vec{v} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n - a_1\vec{u}_1 - \ldots - a_k\vec{u}_k$$

$$b_1\vec{v}_1 + \ldots + b_n\vec{v}_n + a_1\vec{u}_1 + \ldots + a_k\vec{u}_k = 0$$

$$T(b_1\vec{v}_1 + \ldots + b_n\vec{v}_n + a_1\vec{u}_1 + \ldots + a_k\vec{u}_k) = T(0)$$

$$T(b_1\vec{v}_1 + \ldots + b_n\vec{v}_n) = 0$$

$$b_1\vec{w}_1 + \ldots + b_n\vec{w}_n = 0$$

$$b_1 = \ldots = b_n = 0$$

$$a_1\vec{u}_1 + \ldots + a_k\vec{u}_k = 0$$

$$a_1 = \ldots = a_k = 0$$

$$\square$$