

Eigenvectors

An eigenvector of a transformation is a vector whose direction is unchanged by that transformation. The transformation merely scales this vector by its corresponding scalar eigenvalue. This can be represented as $Mv = \lambda v$, where λ is the scalar eigenvalue associated with v .

Each eigenvalue of M corresponds to a subspace of the domain called an eigenspace. This is often a single vector (and any multiples thereof), though it can also be a plane or more. The dimension of the eigenspace is called the geometric multiplicity of its eigenvalue. All vectors in any eigenspace of M are considered eigenvectors of M .

To find the eigenvalues λ that satisfy the equation $Ax = \lambda x$, we notice that the zero vector and scalar are trivial solutions of this equation. We want to find values of λ that work for x other than zero. Using some algebraic manipulation, we see that this equation is equivalent to $(A - \lambda I)x = 0$. The kernel/nullspace of the transformation $(A - \lambda I)$ is only zero, unless its determinant is zero. Therefore, we find the determinant of $(A - \lambda I)$ and set it equal to zero, to find λ that have more than just the trivial solution. This determinant is known as the characteristic polynomial, and its roots are the eigenvalues of the matrix.

Diagonalization

If the union of all the eigenspaces spans the domain/codomain, then we can construct an eigenvector-only basis for the domain. When we want to apply the transformation to a vector, we can first break the vector into a linear combination of eigenvectors, transform those, and then convert back into a vector like the one we were originally given.

Why would we want to do this? Well, we notice that transforming eigenvectors is much easier than transforming any ol' vector, because $T(\vec{v}) = \lambda\vec{v}$. For our eigenvector-basis, our transformation is merely a dilation in each direction, without any skewing or rotating or anything else. This means that each coordinate only changes based on its own value, not on any other coordinates' values. In other words, the transformation might look like:

$$T : \begin{cases} T(\vec{v}_1) = \lambda_1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ T(\vec{v}_2) = 0\vec{v}_1 + \lambda_2\vec{v}_2 + 0\vec{v}_3 \\ T(\vec{v}_3) = 0\vec{v}_1 + 0\vec{v}_2 + \lambda_3\vec{v}_3 \end{cases}$$
$$M = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

In other words, diagonalization $A = PDP^{-1}$ can be interpreted as applying P , a shift into the eigenvector basis, the application of modified transformation (now just an eigenvector dilation) D , and then shifting back into the old basis with P^{-1} . P is a set of linearly independent eigenvectors; clearly, this diagonalization is not possible if the eigenvectors cannot span the domain and codomain.