Definitions

Let's start by rigorizing our definition of linear transformations (linear maps).

$T : V \to W$ is a linear transformation if it satisfies the following two properties:

1. Additivity — $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. Homogeneity — $T(a\vec{u}) = aT(\vec{u})$

V is called the domain, and W the codomain. Define the image (or range) of T as im $T = \{T(\vec{u}) : \vec{u} \in V\}$, and the kernel of T as ker $T = \{\vec{u} \in V : T(\vec{u}) = 0\}$.

Vector Spaces

We are going to focus on linear transformations whose domain and codomain are $\mathbb{R}^n$. We have shown that $\mathbb{R}^n$ is a vector space. Again, a vector space V must have the following properties:

1. Vector addition is associative.
2. Vector addition is commutative.
3. Vector addition has an identity element.
4. Vector addition has an inverse element.
5. Distributivity holds for scalar multiplication over vector addition.
6. Distributivity holds for scalar multiplication over addition between scalars.
7. Scalar multiplication is compatible with multiplication between scalars.
8. Scalar multiplication has an identity element.
9. Vector addition is closed.
10. Scalar multiplication is closed.

It is notable that all vectors in a vector space can be written as a linear combination of vectors; those vectors are said to span the space. If the vectors are linearly independent, they are called a basis for that space. The dimension of a vector space is the number of vectors in a basis.