Contractibility + transport $\Leftrightarrow$ J

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In MLTT, we usually define the identity type as a reflexive relation satisfying J:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M =_A N \ \mathsf{type}} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{refl}_M : M =_A M}$$

$$\frac{\begin{array}{c}\Gamma \vdash M : A \qquad \Gamma \vdash N : A \qquad \Gamma \vdash \alpha : M =_A N \\ \Gamma, x : A, y : A, p : x =_A y \vdash C \ \mathsf{type} \qquad \Gamma \vdash c : C[M/x, M/y, \mathsf{refl}_M/p]\end{array}}{\Gamma \vdash \mathsf{J}[C](\alpha, c) : C[M/x, N/y, \alpha/p]}$$

However, in the presence of sigma types, we can equivalently define the identity type as a reflexive relation such that (1) the based path space is contractible, and (2) dependent types satisfy a lifting property over paths:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M =_A N \ \mathsf{type}} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{refl}_M : M =_A M}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A \qquad \Gamma \vdash \alpha : M =_A N}{\Gamma \vdash \mathsf{contr}(N, \alpha) : (M, \mathsf{refl}_M) =_{\Sigma y:A.(M =_A y)} (N, \alpha)} \qquad \frac{\begin{array}{c}\Gamma \vdash M : A \qquad \Gamma \vdash N : A \qquad \Gamma \vdash \alpha : M =_A N \\ \Gamma, x : A \vdash C \ \mathsf{type} \qquad \Gamma \vdash c : C[M/x]\end{array}}{\Gamma \vdash \mathsf{transport}[C](\alpha, c) : C[N/x]}$$

(I learned this fact from Steve Awodey, who noticed it in 2009.) It is simple to derive contr and $\mathsf{transport}$ from J. To achieve the converse, we start by uncurrying C:

$$\frac{\Gamma, x : A, y : A, p : x =_A y \vdash C \ \mathsf{type}}{\Gamma, x : A, p' : (\Sigma y{:}A.x =_A y) \vdash C[x/x, \pi_1(p')/y, \pi_2(p')/p] \ \mathsf{type}}$$

Call this type C'. Properties of substitution guarantee that:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A \qquad \Gamma \vdash \alpha : M =_A N}{\Gamma \vdash C[M/x, N/y, \alpha/p] \equiv C'[M/x, (N, \alpha)/p'] \ \mathsf{type}}$$

Given the inputs to the J rule, we can then construct the term:

$$\frac{\begin{array}{c}\Gamma \vdash (M, \mathsf{refl}_M) : \Sigma y{:}A.M =_A y \\ \Gamma \vdash (N, \alpha) : \Sigma y{:}A.M =_A y \\ \Gamma \vdash \mathsf{contr}(N, \alpha) : (M, \mathsf{refl}_M) =_{\Sigma y:A.(M =_A y)} (N, \alpha) \\ \Gamma, p' : (\Sigma y{:}A.M =_A y) \vdash C'[M/x, p'/p'] \ \mathsf{type} \\ \Gamma \vdash c : C'[M/x, p'/p'][(M, \mathsf{refl}_M)/p']\end{array}}{\Gamma \vdash \mathsf{transport}[C'[M/x, p'/p']](\mathsf{contr}(N, \alpha), c) : C'[M/x, p'/p'][(N, \alpha)/p']}$$

whose type is exactly the same as that of $\mathsf{J}[C](\alpha, c)$.

Now let us consider β laws for these rules. The J rule satisfies:

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A, y : A, p : x =_A y \vdash C \ \mathsf{type} \qquad \Gamma \vdash c : C[M/x, M/y, \mathsf{refl}_M/p]}{\Gamma \vdash \mathsf{J}[C](\mathsf{refl}_M, c) \equiv c : C[M/x, M/y, \mathsf{refl}_M/p]}$$

If we define contr and $\mathsf{transport}$ from J, we get the derived β laws:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{contr}(M, \mathsf{refl}_M) \equiv \mathsf{refl}_{(M, \mathsf{refl}_M)} : (M, \mathsf{refl}_M) =_{\Sigma y:A.(M =_A y)} (M, \mathsf{refl}_M)}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A \vdash C \ \mathsf{type} \qquad \Gamma \vdash c : C[M/x]}{\Gamma \vdash \mathsf{transport}[C](\mathsf{refl}_M, c) \equiv c : C[M/x]}$$

Therefore, if we were to take contr and $\mathsf{transport}$ as primitive, it would be reasonable, or at least consistent, to require these β laws.

Now we can check that these two β laws imply that J, derived from contr and $\mathsf{transport}$, has the derived β law we expect:

$$\frac{\begin{array}{c}\Gamma \vdash M : A \\ \Gamma, p' : (\Sigma y{:}A.M =_A y) \vdash C'[M/x, p'/p'] \ \mathsf{type} \\ \Gamma \vdash c : C'[M/x, p'/p'][(M, \mathsf{refl}_M)/p']\end{array}}{\Gamma \vdash \mathsf{transport}[C'[M/x, p'/p']](\mathsf{contr}(M, \mathsf{refl}_M), c) \equiv c : C'[M/x, p'/p'][(M, \mathsf{refl}_M)/p']}$$

This result relies on the β laws for both contr and $\mathsf{transport}$. Note that the type of c is definitionally equal, by properties of substitution, to $C[M/x, M/y, \mathsf{refl}_M/p]$.